- Finish: Kirchberg $\Rightarrow$ Tsirelson
- Start the model theory route

Reminder: $\mathbb{F}(k,n) = \underbrace{\mathbb{Z}_n * \cdots * \mathbb{Z}_n}_{k \text{ times}}$

$e^x_1, \ldots, e^x_n$ spectral projections for the generator of the x^{th} copy of $\mathbb{Z}_n$

A PVM in $C^*(\mathbb{F}(k,n))$

POVMs in $\mathcal{H}$ $\equiv$ ucp maps $\Phi: C^*(\mathbb{F}(k,n)) \to \mathcal{B}(\mathcal{H})$

$\downarrow$

Families $(A^x)_{x \in [k]}$ $\quad \Phi(e^x_a) = A^x_a$

Thm If $p \in [0,1]^{k^2n^2}$, then:

① $p \in C_{qa}(k,n) = C_q(k,n)$ iff there is a state ϕ on $C^*(\mathbb{F}(k,n)) \otimes_{min} C^*(\mathbb{F}(k,n))$ s.t. $p(a,b|x,y) = \phi(e^x_a \otimes e^y_b)$

② Same for $C_{qc}(k,n)$ and $\otimes_{max}$.

Fact If A is a C^*-alg and $a \in A_+$, then $\|a\| = \sup\{\phi(a) : \phi \text{ state on } A\}$.

Given game $\mathcal{G}$, define

$$\eta_{\mathcal{G}} \in C^*(\mathbb{F}(k,n)) \odot C^*(\mathbb{F}(k,n)) :$$

$$\eta_{\mathcal{G}} = \sum_{(x,y)\in[k]^2} \pi(x,y) \sum_{(a,b)\in[n]^2} D(x,y,a,b)\,\underbrace{(e_a^x \otimes e_b^y)}$$

$$\therefore \quad val^*(\mathcal{G}) = \|\eta_{\mathcal{G}}\|_{min}$$

$$val^{co}(\mathcal{G}) = \|\eta_{\mathcal{G}}\|_{max}$$

effectively approx from above:

$$C^*(\mathbb{F}(k,n)) \otimes_{max} C^*(\mathbb{F}(k,n)) \cong C^*(\underbrace{\mathbb{F}(k,n) \times \mathbb{F}(k,n)}_{\text{finitely presented}})$$

Cor If $(C^*(\mathbb{F}(k,n)), C^*(\mathbb{F}(k,n)))$ is a nuclear pair, then Tsirelson is true.

Kirchberg: $(C^*(\mathbb{F}_k), C^*(\mathbb{F}_k))$ nuclear pair.
Close enough...

Pf of Thm

① ($\Rightarrow$) WLOG, $p \in C_{qs}(k,n)$.
Have Hilbert spaces H_A, H_B,
POVMs $(A^x)_{x\in[k]}$, $(B^x)_{x\in[k]}$ on H_A, H_B,
unit vector $\xi \in H_A \otimes H_B$ s.t.
$p(a,b|x,y) = \langle (A^x_a \otimes B^y_b)\xi, \xi\rangle$.
ucp maps $\Phi_A : C^*(\mathbb{F}(k,n)) \to \mathcal{B}(H_A)$
$\Phi_B : C^*(\mathbb{F}(k,n)) \to \mathcal{B}(H_B)$
$\therefore$ ucp $\Phi_A \otimes \Phi_B : C^*(\mathbb{F}(k,n)) \otimes C^*(\mathbb{F}(k,n))$
$\longrightarrow \mathcal{B}(H_A) \otimes_{\min} \mathcal{B}(H_B)$

$= \mathcal{B}(H_A \otimes H_B)$

Define, for $\eta \in C^*(\mathbb{F}(k,n)) \otimes C^*(\mathbb{F}(k,n))$,

$\phi(\eta) := \langle (\Phi_A \otimes \Phi_B)(\eta)\, \xi, \xi \rangle$

This ϕ is as desired.

($\Leftarrow$) Have state ϕ on

$C^*(\mathbb{F}(k,n)) \otimes_{\min} C^*(\mathbb{F}(k,n))$.

$C^*(\mathbb{F}(k,n)) \subseteq \mathcal{B}(H)$

$\therefore C^*(\mathbb{F}(k,n)) \otimes_{\min} C^*(\mathbb{F}(k,n)) \subseteq \mathcal{B}(H \otimes H)$

Extend ϕ to a state, also called ϕ, on $\mathcal{B}(H \otimes H)$.

Approximate ϕ by states of the form

$\sum_{i=1}^{n} \lambda_i \langle \cdot\, \xi_i, \xi_i \rangle$, ξ_i unit vectors in $H \otimes H$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$.

when applied to $e^x_a \otimes e^y_b$ give correlations in $C_{qs}(k,n)$.

Use $C_{qa}(k,n)$ convex, closed. $\square$

Enter model theory!

Formal language for talking about tracial vNas.

Quantifier-free formula: $u(\varphi_1(\vec{x}),\ldots,\varphi_m(\vec{x}))$, where each φ_i is of the form

atomics

$tr(p_i(\vec{x}))$, $p_i(\vec{x})$ $*$-poly. in $\vec{x}$

noncommuting,

$u: \mathbb{R}^m \to \mathbb{R}$ continuous function

"connectives"

If $\varphi(\vec{x})$ is a formula, then so are $\sup_{\|x_i\|\leq m} \varphi(\vec{x})$, $\inf_{\|x_i\|\leq m} \varphi(\vec{x})$.

Interpretations: If $\varphi(\vec{x})$ is a formula, (M,τ) tracial vNa, $\vec{a}$ tuple from M, $\varphi(\vec{a})^M$ = result of "plugging in" $\vec{a}$ into $\vec{x}$
= real #

ex $\varphi(x)$ was $\displaystyle\sup_{\|y\|\leq 1} \overbrace{\|xy-yx\|_2}^{\text{qf formula}}$

For (M,τ) tracial vNa, $a\in M$,
$\varphi(a)^M = 0$ iff $a\in Z(M)$.

Sentence: No free variables.

ex $\sigma = \displaystyle\sup_{\|x\|\leq 1} \varphi(x)$, φ as above.
$\sigma^M = 0$ iff M abelian.

- Tracial vNas are axiomatizable as are $\underbrace{\text{II}_1 \text{ factors}}_{T_{\text{II}_1} \text{ axioms}}$.

CEP & Model Theory

Universal sentence: $\sigma = \sup_{\|\vec{x}\| \le 1} \varphi(\vec{x})$ q.f.

Universal theory of (M,τ): function

$$\mathrm{Th}_\forall(M,\tau) : \{\text{universal sentences}\} \to \mathbb{R}$$
$$\sigma \mapsto \sigma^{(M,\tau)}$$

Notice: If $(M,\tau) \hookrightarrow (N,\tau)^{\mathcal{U}}$, then
$$\mathrm{Th}_\forall(M,\tau) \le \mathrm{Th}_\forall(N,\tau)$$

Model theory 101: Converse is true:
If $\mathrm{Th}_\forall(M,\tau) \le \mathrm{Th}_\forall(N,\tau)$, then
$$(M,\tau) \hookrightarrow (N,\tau)^{\mathcal{U}}.$$

Since $\mathcal{R}$ embeds into any II_1 factor, get:

Thm CEP is equivalent to:
$\mathrm{Th}_\forall(M) = \mathrm{Th}_\forall(\mathcal{R})$ for all II_1 factors M.

Thm (G. and Hart) If CEP is true, then $Th_r(\mathcal{R})$ is computable: there is an algorithm so that, upon inputs σ, ε, returns an interval $I \subseteq \mathbb{R}$, $|I| < \varepsilon$, with $\sigma^{\mathcal{R}} \in I$.

Rmk: Can always approximate such σ from below effectively. $\mathcal{R} = \overline{\bigcup M_{2^n}(\mathbb{C})}$

Pf of Thm: Key: There is a Soundness and Completeness Thm for Continuous Logic (Ben Yaacov / Pederson)

$$\underbrace{\sup\{\sigma^M : M \text{ II}_1 \text{ factor}\}}_{\substack{\| \\ \sigma^{\mathcal{R}}}} \underset{\substack{M \hookrightarrow \mathcal{R}^{\mathcal{U}} \\ \text{assuming CEP}}}{} = \inf\{r \in \mathbb{Q}^{>0} : \underbrace{T_{\text{II}_1}}_{\text{effectively enumerable}} \vdash \sigma \leq r\}$$

Can effectively enumerate r's for which $T_{\mathrm{II}_1} \vdash \sigma \le r$.

This algorithm gives effective upper bounds. ☐

Thm (G. & Hart) $MIP^* = RE \Rightarrow \mathrm{Th}_\forall(\mathcal{R})$ is not computable.

$\mathcal{R} = \overline{\bigcup M_{2^n}(\mathbb{C})}$

Idea: $\mathrm{val}^*(\mathcal{G})$ should be expressible as $\sigma_{\mathcal{G}}^{\mathcal{R}}$ for some universal sentence σ.

Question: What $\sigma_{\mathcal{G}}$?

Def $p \in C_{qa}(k,n)$ is *synchronous* if, $\forall x$ $p(a,b \mid x,x) = 0$ if $a \neq b$.

$C_{qa}^s(k,n)$ = set of synchronous qa-strat.

Rmk: $s\text{-}\mathrm{val}^*(\mathcal{G}) = \sup_{p \in C_{qa}^s(k,n)} \mathrm{val}(\mathcal{G}, p)$

$s\text{-}val^*(\mathcal{G}) \leq val^*(\mathcal{G})$

But: The games in $MIP^* = RE$ are s.t. if $\mathcal{M}$ halts, then $s\text{-}val^*(\mathcal{G}_\mathcal{M}) = 1$.

Thm (Kim, Paulsen, Schafhauser)

$p \in C^s_{qa}(k,n)$ iff there are projections $e^x_1, \ldots, e^x_n$ in $\mathcal{R}^u$ (for each $x \in [k]$) with $\sum_{a=1}^n e^x_a = 1$ and $p(a,b|x,y) = \tau(e^x_a e^y_b)$ ← atomic formula

τ unique trace on $\mathcal{R}^u$

$$\therefore\ s\text{-}val^*(\mathcal{G}) = \Bigg(\boxed{\sup_{\vec{e}}}^{\,?} \underbrace{\sum_{(x,y) \in [k]} \pi(x,y) \sum_{(a,b) \in [n]} D(x,y,a,b)\, \tau(e^x_a e^y_b)}_{q.f.} \Bigg)^{\mathcal{R}^u}$$

Definable sets

Close!

If Γ is any ICC amenable group (e.g. $\Gamma = \bigcup_{n=1}^{\infty} S_n$), then $L(\Gamma) \cong \mathcal{R}$.

$\mathcal{R}$ is an e.c. model of $Th_\forall(\mathcal{R})$.